MATH 175: Final Exam Review for Pre-calculus

In order to prepare for the final exam, you need too be able to work problems involving the following topics:

1. Can you graph rational functions by hand after algebraically or numerically finding their asymptotes or holes, if any and then verify your results using a graphing utility?

**Prob.1.1.** Sketch a complete *labeled* graph for the following function after listing any (and all) vertical asymptotes, holes, and horizontal asymptotes.

\[ f(x) = \frac{(x+5)^2(x-1)}{(x^2+6x+5)(x+5)} \]

2. Can you determine whether a function has an inverse function and, if possible, find it?

**Prob. 2.1.** Give a reason why you expect the function \( f(x) = \sqrt[3]{x^3 + x - 2} \) to have an inverse \( f^{-1}(x) \). Sketch the graph \( f(x) \) and \( f^{-1}(x) \) on the same set of axes, clearly labeling each. (use ZDecimal for the axes.)

**Prob.2.2** Why doesn’t the function \( f(x) = 3(x+7)^2 - 2 \) have an inverse function? Give a reason (which could be in the form of a labeled sketch with comments).

3. Can you define and graph the basic exponential and logarithmic functions and their transformations by hand?

**Prob.3.1** Sketch a labeled graph of the function \( f(x) = \begin{cases} e^{x+1} & \text{if } x < 1 \\ \ln(x) & \text{if } x \geq 1 \end{cases} \)

4. Can you solve exponential and logarithmic equations algebraically and/or graphically using the laws of logarithms where appropriate?

**Prob.4.1** Solve the following equations for \( x \). Show your steps and give *exact* answers.

a. \( 7^x = 5^{x+3} \)

b. \( \log_9(3^{x^2}) = -2 \)

5. Can you find the exact values of the trigonometric functions using the unit circle, point in the plane, and the right triangle definitions of the trigonometric functions?

**Prob.5.1.** Sketch a unit circle with the given \( \theta \) values marked on it.

Then find the *exact* functional value requested.

a. Find \( \sin(\theta) \) if \( \theta = 5\pi/4 \)

b. Find \( \csc(\theta) \) if \( \theta = -5\pi/6 \)
Prob.5.2. If \( \sin(\theta) = \frac{3}{4} \) and \( \cos(\theta) < 0 \):

(a.) What Quadrant is the terminal side of \( \theta \) in?

(b.) Give the **exact** values of \( \cos(\theta) \) and of \( \tan(\theta) \).

\[ \cos(\theta) = \quad \tan(\theta) = \]

6. Can you use the basic trigonometric identities to prove other identities and to find exact values of the other trigonometric functions given the exact value of one function?

Prob.6.1 Complete the following identities with an equivalent expression.

a. \( \tan(-\theta) = \quad \)

b. \( \cos(A)\sin(B) - \sin(A)\cos(B) = \quad \)

c. \( 1 + \tan^2(\theta) = \quad \)

d. \( \cos(2A) = \quad \)

e. \( 2\cos(A)\sin(A) = \quad \)

f. \( \cos^2 A = \frac{1}{2} \left( 1 \quad \right) \)

g. \( \cos^2(15^\circ) + \sin^2(15^\circ) = \quad \)

Prob.6.2. Use the appropriate sum or difference formula to find the **exact** value of:

(Show your steps.)

a. \( \cos\left(\frac{5\pi}{12}\right) = \quad \)

b. \( \sin(12^\circ)\cos(48^\circ) + \sin(48^\circ)\cos(12^\circ) \)

Prob.6.3 Prove the identity, show all work.

\[ \tan(\alpha) + \sec(\alpha) = \frac{1}{\sec(\alpha) - \tan(\alpha)} \]

7. Can you graph basic trigonometric functions and transformations of the sine and cosine function by hand?

Prob.7.1 The simple harmonic motion of a spring that is initially displaced from its rest position is given by the equation \( y = 5\cos\left(\frac{\pi x}{2}\right) \) where \( x \) is in seconds and \( y \) is in centimeters.

a. Sketch a labeled graph of this equation for time \( x = 0 \) to \( x = 8 \).

b. What is the maximum displacement?

c. Mark with an asterisk and give the coordinates for each time that the spring is 4 cm below the rest position.
8. Can you solve trigonometric equations both algebraically and graphically?

**Prob.8.1** Find all real numbers x that satisfy: \(2 \sin^2(x) - 3 \sin(x) = 2\)

9. Can you solve right and oblique triangles?

**Prob.9.1** Which of the following statements about the angle \(\theta\) is true?

a. \(\sin(\theta) = 3/4\)
b. \(\cos(\theta) = 5/4\)
c. \(\tan(\theta) = 3/5\)
d. \(\sin(\theta) = 4/5\)
e. \(\sin(\theta) = 4/3\)

**Prob.9.2** Find the length of side \(h\) in the triangle, where angle A measures 40° and the distance from C to A is 25.

10. Can you evaluate inverse sine, inverse cosine, and inverse tangent function values exactly when possible and, if not possible, approximate using a graphing utility?

**Prob.10.1**

a. What is the domain and range for the function \(f(x) = \cos^{-1}(x)\)?
   b. Give exact values in radians of \(\cos^{-1}(\sqrt{2}/2)\).

**Prob.10.2**

a. What is the domain and range for the function \(f(x) = \tan^{-1}(x)\)?
   b. Give exact values in radians of \(\tan^{-1}(\sqrt{3})\).

**Prob.10.3**

a. What is the domain and range for the function \(f(x) = \sin^{-1}(x)\)?
   b. Give exact values in radians of \(\sin^{-1}(\sin(8\pi/3))\).

**Prob.10.4**

a. Give exact values in radians of \(\cos^{-1}(\sin(5\pi/3))\).
   b. Give the exact value of \(\tan(\cos^{-1}(-1/3))\).

**Prob.10.5** Find all real numbers x that satisfy the equation: \(\tan(x) = 3\)
11. Can you solve applications involving exponential, logarithmic, and trigonometric functions algebraically or by using a graphing utility?

**Prob.11.1** The half-life of calclinium is 100 days. If you start with 100 milligrams, write down the expression for the amount of calclinium remaining after t days. Use your result to find the time when 33% of the calclinium you started with remained.

**Prob.11.2**

a. Define all 5 variables in the formula: 
   \[ A = P \left( 1 + \frac{r}{n} \right)^{nt} \].

b. Use this formula and algebra to determine how long it would take an investment of $1500 to grow to $5000 at an annual interest rate of 7.5% and weekly compounding. An answer without the required steps will not receive full credit.

**Prob.11.3** The annual rate of growth of the world’s population in 2005 was \( k = 1.15\% \). The world’s population in 2005 was 6,451,058,790.

a. Let \( t = 0 \) represent 2005 and use the uninhibited growth model to predict the world’s population in 2015.

b. Using this model determine how long it would take for the population to double.

**Prob.11.4** A pilot in a plane at an altitude of 22,000 feet observes that the angle of depression to a nearby airport is 26°. How many miles is the airport from a point on the ground directly below the plane? A labeled sketch of the situation is required.

12. Can you convert rectangular coordinates to polar coordinates and vice versa?

Can you convert rectangular equations to polar equations and vice versa?

**12.1.** Convert the polar coordinates \( (\sqrt{3}, \frac{3\pi}{4}) \) to rectangular coordinates.

**12.2.** Convert the rectangular coordinates \((-6.2, -3)\) to polar coordinates with \( \theta \in [0, 2\pi) \).

**12.3.** Write the equations of the following polar curves in rectangular coordinates.

a. \( r = 3 \)

b. \( \theta = 3.5 \)

c. \( r = 4\sin\theta \)

**12.4.** Write the equations of the following rectangular coordinate curves in polar coordinates.

a. \( x^2 + y^2 = 16 \)

b. \( (x-3)^2 + y^2 = 9 \)

c. \( y = x \)
Answers to the Final Exam Review questions for Math 175.

Prob.1.1.

\[ f(x) = \frac{(x + 5)^2(x - 1)}{(x^2 + 6x + 5)(x + 5)} = \frac{(x + 5)^2(x - 1)}{(x + 1)(x + 5)(x + 5)} = \frac{x - 1}{x + 1} \]

Hole at (-5, 3/2), vertical asymptote: x = -1, horizontal asymptote: y=1

Prob. 2.1.

Use your graphing calculator to graph \( f(x) = \sqrt[3]{x^3 + x - 2} \) and \( y = x \), then use "DRAWINVERSE" from the DRAW menu to see the graph of \( f^{-1}(x) \)

Prob.2.2 \( y = f(x) = 3(x + 7)^2 - 2 \) is not 1 – 1 because \( x = -7 \pm \sqrt{(y + 2)/3} \).
Prob. 3.1 \( f(x) = \begin{cases} e^{x+1} & \text{if } x < 1 \\ \ln(x) & \text{if } x \geq 1 \end{cases} \)

\[ y = e^{x+1} \text{ if } x < 1 \quad (1, e^2) \quad y = \ln(x) \text{ if } x \geq 1 \]

Prob. 4.1 (a.) \( x = 3\ln(5)/\ln(7/5) \) \hspace{1cm} (b.) \( x = \pm 2 \) \hspace{1cm} (c.) no solution

Prob. 5.1. (a.) \( \sin(5\pi/4) = -\sqrt{2}/2 \) \hspace{1cm} (b.) \( \csc(-5\pi/6) = -2 \)

Prob. 5.2. (a.) Quadrant II \hspace{1cm} (b.) \( \cos(\theta) = -\sqrt{7}/4, \tan(\theta) = -3\sqrt{7}/7 \)

Prob. 6.1 (a.) \( -\tan(\theta) \) \hspace{1cm} (b.) \( \sin(B - A) = -\sin(A - B) \)
\hspace{1cm} (c.) \( \sec^2(\theta) \) \hspace{1cm} (d.) \( \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A \)
\hspace{1cm} (e.) \( \sin(2A) \) \hspace{1cm} (f.) \( 1 + \cos(2A) \)/2 \hspace{1cm} (g.) 1

Prob. 6.2 (a.) \( \sqrt{2}(\sqrt{3} + 1)/4 \) \hspace{1cm} (b.) \( \sin(60^\circ) = \sqrt{3}/2 \)

Prob. 6.3
\[
\frac{1}{\sec(\alpha) - \tan(\alpha)} = \frac{\cos(\alpha)}{(1 - \sin(\alpha))} \frac{(1 + \sin(\alpha))}{(1 + \sin(\alpha))} = \frac{\cos(\alpha) + \cos(\alpha)\sin(\alpha)}{\cos^2(\alpha)} = \frac{1}{\cos(\alpha)} + \frac{\sin(\alpha)}{\cos(\alpha)} = \tan(\alpha) + \sec(\alpha)
\]
Prob. 7.1 \[ y = 5 \cos \left( \frac{\pi x}{2} \right) \]

a. Graph
b. max displacement is 5 cm

c. \[ x_1 \approx 1.59 \quad x_2 \approx 2.41 \quad \text{and} \quad x_3 \approx 5.59 \quad x_4 \approx 6.41 \]

Prob. 8.1 \[ 2 \sin^2(x) - 3 \sin(x) - 2 = (2 \sin(x) + 1)(\sin(x) - 2) = 0. \]
So \[ x = \frac{7\pi}{6} \pm 2k\pi \quad \text{or} \quad x = \frac{11\pi}{6} \pm 2k\pi \quad \text{where} \quad k = 0, 1, 2, \ldots \]

9. Solve right and oblique triangles.

Prob. 9.1 (d.) is true, i.e. \( \sin(\theta) = 4/5 \)

Prob. 9.2 \( h = 25/\cos(40^\circ) \approx 32.6352 \)

Prob. 10.1 a. The domain is \([-1,1]\) and the range is \([0,\pi]\)
b. \( \cos^{-1}(\sqrt{2}/2) = \pi/4 \)

Prob. 10.2 a. The domain is \((-\infty, \infty)\) and the range is \((-\pi/2, \pi/2)\)
b. \( \tan^{-1}(\sqrt{3}) = \pi/3 \)

Prob. 10.3 a. The domain is \([-1,1]\) and the range is \([-\pi/2, \pi/2]\)
b. \( \sin^{-1}(\sin(8\pi/3)) = \pi/3 \)

Prob. 10.4 a. \( \cos^{-1}(\sin(5\pi/3)) = 5\pi/6 \)  
  b. \( \tan(\cos^{-1}(-1/3)) = -2\sqrt{2} \)

Prob. 10.5 \( \tan^{-1}(3) \pm n\pi \approx 1.24905 \pm n\pi \) with \( n = 1, 2, 3, \ldots \)

Prob. 11.1 The amount after \( t \) days is \( A(t) = 100e^{(-t \ln(2)/100)} \approx 100e^{-0.0069t} \) and it will take about 160 days until only 33% is left.
**Prob. 11.2**  
\[ A = P \left( 1 + \frac{r}{n} \right)^{(nt)} \]  
A is the amount after \( t \) years, \( P \) is the Principal (original amount invested), \( r \) is the interest rate written as a decimal, \( n \) is the number of compounding periods per year, and \( t \) is the number of years.

b. Solve \( 5000 = 1500 \left( 1 + \frac{0.075}{52} \right)^{(52t)} \) for \( t \). Then \( t = \frac{\ln(10/3)}{(52 \ln(1 + 0.075/52))} \approx 16 \text{ yrs.} \)

**Prob. 11.3**  
a. The population model is \( P(t) = 6,451,058,790e^{0.0115t} \).  
\( P(10) = 7,237,271,501 \)

b. The doubling time predicted for this model is \( t_d = \frac{\ln(2)}{0.0115} \approx 60.3 \text{ years} \).

**Prob. 11.4**  
The plane is \( \frac{22000}{\tan(26^\circ)} \) feet \( \approx 45,106.7 \text{ feet or approximately 8.54 miles from the airport.} \)

**12.1.** \((-\sqrt{6}/2, \sqrt{6}/2)\)

**12.2.** \((\sqrt{47.44}, \tan^{-1}(3/6.2) + \pi)\)

**12.3.**  
a. \( x^2 + y^2 = 9 \)  
b. \( y = (\tan 3.5)x \)  
c. \( x^2 + (y - 2)^2 = 4 \)

**12.4.**  
a. \( r = 4 \)  
b. \( r = 6\cos \theta \)  
c. \( \theta = \frac{\pi}{4} \)